

## Exercise 23

Use a calculator to evaluate the line integral correct to four decimal places.

$$\int_C \mathbf{F} \cdot d\mathbf{r}, \quad \text{where } \mathbf{F}(x, y) = \sqrt{x+y}\mathbf{i} + (y/x)\mathbf{j} \text{ and } \mathbf{r}(t) = \sin^2 t \mathbf{i} + \sin t \cos t \mathbf{j}, \quad \pi/6 \leq t \leq \pi/3$$

### Solution

With the given parameterization in  $t$ , the line integral becomes

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\pi/6}^{\pi/3} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_{\pi/6}^{\pi/3} \left\langle \sqrt{x(t)+y(t)}, \frac{y(t)}{x(t)} \right\rangle \cdot \frac{d}{dt} \langle \sin^2 t, \sin t \cos t \rangle dt \\ &= \int_{\pi/6}^{\pi/3} \left\langle \sqrt{\sin^2 t + \sin t \cos t}, \frac{\sin t \cos t}{\sin^2 t} \right\rangle \cdot \langle 2 \sin t \cos t, \cos^2 t - \sin^2 t \rangle dt \\ &= \int_{\pi/6}^{\pi/3} \left\langle \sqrt{\sin^2 t + \sin t \cos t}, \frac{\cos t}{\sin t} \right\rangle \cdot \langle 2 \sin t \cos t, \cos^2 t - \sin^2 t \rangle dt \\ &= \int_{\pi/6}^{\pi/3} \left[ 2 \sin t \cos t \sqrt{\sin^2 t + \sin t \cos t} + \frac{\cos t}{\sin t} (\cos^2 t - \sin^2 t) \right] dt \\ &= \int_{\pi/6}^{\pi/3} \left( \sin 2t \sqrt{\sin^2 t + \frac{1}{2} \sin 2t} + \cot t \cos 2t \right) dt. \end{aligned}$$

Let  $f(t)$  represent the integrand.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\pi/6}^{\pi/3} f(t) dt$$

Use Simpson's rule with  $n = 10$ .

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &\approx \frac{\Delta t}{3} [f(t_0) + 4f(t_1) + 2f(t_2) + 4f(t_3) + 2f(t_4) + 4f(t_5) \\ &\quad + 2f(t_6) + 4f(t_7) + 2f(t_8) + 4f(t_9) + f(t_{10})] \\ &\approx \frac{\frac{\pi}{3} - \frac{\pi}{6}}{3(10)} \left[ f\left(\frac{\pi}{6}\right) + 4f\left(\frac{11\pi}{60}\right) + 2f\left(\frac{\pi}{5}\right) + 4f\left(\frac{13\pi}{60}\right) + 2f\left(\frac{7\pi}{30}\right) + 4f\left(\frac{\pi}{4}\right) \right. \\ &\quad \left. + 2f\left(\frac{4\pi}{15}\right) + 4f\left(\frac{17\pi}{60}\right) + 2f\left(\frac{3\pi}{10}\right) + 4f\left(\frac{19\pi}{60}\right) + f\left(\frac{\pi}{3}\right) \right] \\ &\approx \frac{\frac{\pi}{3} - \frac{\pi}{6}}{3(10)} (31.0789) \\ &\approx 0.542429 \end{aligned}$$